Quadratic Equations

Lessons Overview

- **Introduction to Quadratic Equations**
- **Types of Equations:**
  - Pure
  - Incomplete
  - Perfect Squares
  - Special Trinomials
- **Solving Techniques:**
  - Factoring (ZPP)
  - Completing the Square
  - General Formula
- **Discriminant Cases**
- **Step-by-step Exercises**

Quadratic Equations

Standard Form Equation

A quadratic equation in standard form is written as:

$$ax^2 + bx + c = 0.$$

- a, b, and c are real numbers, with $a \neq 0$ (otherwise it wouldn't be a quadratic equation).
- "Standard form" simply means the terms are ordered by degree.

Example

1. What is the standard form of $4 + x = -x^2$?
2. Is $x^2 + 2 = 0$ in standard form?

Roots of an Equation

Roots of an Equation

To a quadratic equation like $ax^2 + bx + c = 0$, we can associate the polynomial $P(x) = ax^2 + bx + c$.
A solution x_0 is a value that, when substituted into $P(x)$, makes the polynomial equal to zero.

Example

Let's consider the equation $x^2 - 4 = 0$, with associated polynomial $P(x) = x^2 - 4$.
Substituting $x = 2$:

$$P(2) = 2^2 - 4 = 4 - 4 = 0$$

Since $P(2) = 0$, $x = 2$ is a solution of the equation $x^2 - 4 = 0$.

Question: What are the solutions to $x^2 + 2 = 0$?

Square Roots and Quadratic Equation Solutions

Square Root

The square root of a number, such as $\sqrt{9}$, is defined as the positive value whose square gives the original number. For example, $\sqrt{9} = 3$, since $3^2 = 9$. Yet, note that $(-3)^2 = 9$ as well.
We define the square root as the positive value only, to keep the operation well-defined and simple.

Solving Quadratic Equations

When solving an equation like $x^2 = 9$, we look for all numbers whose square is 9.
In this case, there are two solutions: $x = 3$ and $x = -3$, since both 3^2 and $(-3)^2$ equal 9. Unlike the square root, we consider both the positive and negative roots.

Square Roots of Negative Numbers and No Real Solutions

No Real Square Root of Negative Numbers

The square root of a number, such as $\sqrt{81}$, is a number that squared gives the radicand.
However, there is no real square root of a negative number like $\sqrt{-81}$, since no real number squared can yield a negative result.

Quadratic Equations Without Solutions

This means that some quadratic equations have no real solution. For example, the equation $x^2 + 2 = 0$ has no real solutions, because solving it would require computing the square root of -2, which is not a real number.

Number of Solutions of a Quadratic Equation

Number of Solutions

A quadratic equation can have 0, 1, or 2 real solutions:

- 0 if the equation has no real roots,
- 1 if it has a double root,
- 2 if the root involves a positive square root.

Exercise

Determine how many solutions each equation has:

1. $x^2 = 64$
2. $x^2 = -7$
3. $(x + 3)^2 = 0$

Quadratic Equations with One Solution

Perfect Square and Single Solution

A quadratic equation has a unique real solution when the associated polynomial is a perfect square of a binomial. This means it can be written as:

$$(x+a)^2$$

The formula is:

$$(x+a)^2 = x^2 + 2ax + a^2$$

If a quadratic is of this form, it has one solution.

Example

Consider the equation $x^2 + 4x + 4 = 0$.

- This polynomial is the square of $(x+2)$, since:

$$x^2 + 4x + 4 = (x+2)^2$$

- Since $(x+2)^2 = 0$, the only solution is $x = -2$.
- The solution is the value that makes the binomial vanish.

Pure Quadratic Equations

What Are Pure Quadratic Equations?

Pure quadratic equations are a special type of quadratic equation where the linear term is missing. They have the form:

$$ax^2 + c = 0$$

In these equations, the variable appears only as a square. To solve them, just isolate x^2 and find the **two values** that satisfy the equation.

Example: Solving $2x^2 - 8 = 0$

- First, isolate x^2:
$$2x^2 = 8$$
- Then, divide both sides by 2:
$$x^2 = 4$$
- Finally, find the two numbers whose square is 4:
$$x = -2 \vee x = 2 \quad (x = \pm 2)$$

Exercises on Quadratic Equations

Exercises

Solve the following equations:

1. $x^2 - 5 = 0$
2. $4x^2 - 1 = 0$
3. $x^2 + \sqrt{2} - \sqrt{3} = 0$
4. $x^2 - \sqrt{2} + \sqrt{3} = 0$
5. $4x^2 - x(2 - x) + 2(x - 1) = 3$
6. $(x - 1)^2 + (x + 1)^2 = (x + 1)(x - 1) + 5$
7. Which real numbers are equal to their own square?

Solutions to the Exercises

Solutions

1. $x^2 - 5 = 0$
   Solution: $x = \pm\sqrt{5}$
2. $4x^2 - 1 = 0$
   Solution: $x = \pm\frac{1}{2}$
3. $x^2 + \sqrt{2} - \sqrt{3} = 0$
   Solution: $x = \pm\sqrt{\sqrt{3} - \sqrt{2}}$
4. $x^2 - \sqrt{2} + \sqrt{3} = 0$
   No real solution, since $x^2 = \sqrt{2} - \sqrt{3}$ is a square equal to a negative number.
5. $4x^2 - x(2 - x) + 2(x - 1) = 3$
   Simplifies to: $5x^2 - 5 = 3 \Rightarrow 5x^2 = 8 \Rightarrow x = \pm 1$
6. $(x - 1)^2 + (x + 1)^2 = (x + 1)(x - 1) + 5$
   Expands to: $2x^2 + 2 = x^2 + 4 \Rightarrow x^2 = 2$
   Solution: $x = \pm\sqrt{2}$
7. Answer: 0 and 1 are the only real numbers equal to their own square.